Chapter 7

Orbital equations

from geodesics
to
real orbits

So far, we have used only the effective potential to establish the broad relativistic orbital behavior. In this section we will discuss the general orbital equations for Newtonian and relativistic orbits around Schwarzschild black holes.

Circular orbits are still relatively simple, since they have a constant radius and are therefore not influenced by space curvature. Non-circular orbits are pretty complex, because the object moves through curved space for most of the time.

With the equations for $\tilde{L}$, $\tilde{E}$ and $\tilde{V}$, as derived in a previous section, it is only possible to directly solve for r at the turning points of orbits. In this section, we will first look at Newton orbits again and then show how the relativistic solution differs from Newton's.

The approach followed here is a bit of a mixture between the approaches of [Faber] and [MTW]. See the box *Comparison to Faber and MTW orbital equations* on page 116 for some clarification.

Newtonian orbits As we have shown before, Newtonian orbits can be solved by using Kepler's laws. Alternatively, the solution can be found by using Newton's law of gravity and his laws of motion.

By writing down the Newtonian equation of motion for an object moving in a gravitational field and solving it, the following differential equation can be obtained, e.g., [Faber]:

$$\frac{d^2r}{dt^2} = r\frac{d\phi^2}{dt^2} - \frac{GM}{r^2}, \tag{7.1}$$

where ϕ is the orbital angle as measured from the point of closest approach. This gives the rate of change of the radial velocity ($d^2r/dt^2 = v_r/dt$). It equals, as expected, the sum of the positive centrifugal acceleration and the negative gravitational acceleration.

Note that $rd\phi^2/dt^2 = v_t^2/r$, the more well known expression for centrifugal acceleration. It is easy to see that for circular orbits, $d^2r/dt^2 = 0$ and one can directly solve the for constant circular orbital speed, i.e.,

$$v_o = \sqrt{GM/r} \tag{7.2}$$

For purposes of comparison with general relativity later, equation 7.1 can be (laboriously) reworked into

$$\left(\frac{du}{d\phi}\right)^2 = 2\frac{(GM)^2}{\tilde{L}_N^2 c^2}\frac{\tilde{E}_N}{c^2} + 2\frac{(GM)^2}{\tilde{L}_N^2 c^2}u - u^2,$$

where again, $u = GM/(rc^2)$, the dimensionless (inverse) distance parameter that simplifies the mathematics somewhat. Now differentiate u with respect to ϕ and divide by $2\frac{du}{d\phi}$, to get a second order differential equation*

*Note that with $u = \frac{GM}{rc^2}$, $\frac{dr}{d\phi} = \frac{-r^2c^2}{GM}\frac{du}{d\phi}$.

$$\frac{d^2u}{d\phi^2} = \frac{(GM)^2}{\tilde{L}_N^2 c^2} - u, \tag{7.3}$$

which can be solved analytically, e.g., [Faber, p. 193]. The result is the equation for a conic section

$$u = \frac{(GM)^2}{\tilde{L}_N^2 c^2}(1 + e\cos\phi) \ , \tag{7.4}$$

or, restoring r:

$$r = \frac{\tilde{L}_N^2}{GM(1 + e\cos\phi)} \ , \tag{7.5}$$

the same as we had before, when we obtained the orbital equation through the Kepler laws for planetary motion (equation 6.4). We will now investigate how the relativistic orbital equations differ from the Newtonian ones above.

Relativistic orbits Although relativistic orbits can be solved by the same procedure as above, this time through the relativistic equations of motion, it is very complex process. A slightly simpler procedure is through the geodesic equations, where the rate of change of r is obtained relative to proper time τ, e.g., [MTW] eq. (25.16):

$$\left(\frac{dr}{d\tau}\right)^2 + \tilde{V}^2 = \tilde{E}^2, \tag{7.6}$$

where $\tilde{V}$ and $\tilde{E}$ are the relativistic effective potential and the energy parameters. From the definition of the angular momentum parameter ($\tilde{L} = r^2 d\phi/d\tau$), $d\tau$ can be extracted and replaced in equation (7.6), resulting in

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{\tilde{L}^2c^2}(\tilde{E}^2 - \tilde{V}^2). \tag{7.7}$$

Because $\tilde{V}$ is a function of r, this equation is not analytically solvable. It is however very easy to solve numerically (point by point), as will be shown later.

Since we would like to compare the relativistic case with the Newtonian case, it is helpful to differentiate this equation to obtain the equivalent second order differential equation. Again employ $u = GM/(rc^2)$ and writing out $\tilde{V}^2 = (1 - 2GM/(rc^2))(1 + \tilde{L}^2/(r^2c^2))$ in equation (7.7) one gets*

*Note that with $u = \frac{GM}{rc^2}$, $\frac{dr}{d\phi} = \frac{-r^2c^2}{GM}\frac{du}{d\phi}$.

$$\left(\frac{du}{d\phi}\right)^2 = \frac{(GM)^2}{\tilde{L}^2c^2}(\frac{\tilde{E}^2}{c^4} - 1) + 2\frac{(GM)^2}{\tilde{L}^2c^2}u - u^2 + 2u^3, \qquad (7.8)$$

an easily differentiable form. Following the recipe used in the Newtonian analysis, differentiate u with respect to ϕ and divide by $2\frac{du}{d\phi}$, to get the second order orbital equation

$$\frac{d^2u}{d\phi^2} = \frac{(GM)^2}{\tilde{L}^2c^2} - u + 3u^2. \qquad (7.9)$$

When compared to the Newtonian orbital equation (7.3), we see that apart from the constants $\tilde{L}_N$ and $\tilde{L}$ that differ, there is an additional 'relativistic' term, $3u^2$. It is immediately obvious that, far from mass M, where $3u^2$ is small compared to the other terms, the equation approaches the Newtonian solution. The extra term makes the equation analytically unsolvable though.

This fact becomes apparent when, after solving any one of equations (7.7), (7.8) or (7.9) numerically, one finds that closed orbits do not repeat themselves—they precess in the direction of normal orbital movement.

This precession becomes increasingly more complex as the orbit comes closer to the black hole. This precession was first observed as the perihelion shift of Mercury, as will be discussed in the next chapter.

Numerical solution of orbital equations By the theorem of differentials, equation (7.7) may be written as

$$dr = \pm\frac{dr}{d\phi}d\phi = \pm\frac{r^2}{c\tilde{L}}\sqrt{\tilde{E}^2 - \tilde{V}^2}\; d\phi, \qquad (7.10)$$

still in geometric units, where $c, G = 1$. Choosing a small constant positive angular increment $\Delta\phi$, we can approximate the orbital equation by

$$\Delta r \cong \pm\frac{r^2}{c\tilde{L}}\sqrt{\tilde{E}^2 - \tilde{V}^2}\; \Delta\phi, \qquad (7.11)$$

and then numerically integrate to find the radius r_j for any orbital angle $\phi_j = \sum \Delta\phi$.

This equation does cause a few hassles though. Firstly, it does not 'know' in which direction r is changing, so one must find a way of telling the algorithm. More seriously, at the periastron and apastron, $\tilde{E} = \tilde{V}$ so that

$dr/d\phi = 0$, i.e., $\Delta r = 0$ for any $\Delta\phi$. Straight forward numerical integration then 'locks up' and r remains constant for varying ϕ (an anomalous circular orbit).

There are various ways to overcome these problems, but it is easier to use the second order differential equation (7.9). It overcomes the problems more readily, because there is no square root taken and $\frac{d^2u}{d\phi^2}$ is non-zero at the periastron and apastron. A simple method of numerically solving equation (7.9) is to recognize that

$$\Delta u_n \cong \Delta u_{n-1} + \frac{d^2u}{d\phi^2}\Delta\phi^2, \qquad (7.12)$$

where Δu_{n-1} is the previously calculated Δu in the numerical integration. Fortunately, Δu_{n-1} and $\frac{d^2u}{d\phi^2}$ cannot become zero at the same time, so the numerical integration will not 'lock up', except in the highly unlikely event that $\Delta u_{n-1} = -\frac{d^2u}{d\phi^2}\Delta\phi^2$ (identically) occurs. To guard against this, one must check for the event and give the orbiter the tiniest of 'nudges' in the right direction.

One remaining hassle is that the algorithm does require knowledge of the 'previous value of Δu_n' before the integration loop starts. This can be overcome if the orbit is started at periapsis or apoapsis, where $\Delta u_{n-1} \cong -\Delta u_n$, so that

$$\Delta u_{n-1} \cong \tfrac{1}{2}\frac{d^2u}{d\phi^2}\Delta\phi^2. \qquad (7.13)$$

Hence, for the next round, one knows the problematic first Δu_{n-1}.

This method was used to plot the orbit in figure 7.1, using the algorithm shown in box *Numerical solution of the second order orbital equation* on page 114.

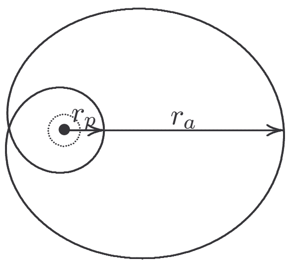

Figure 7.1: A relativistic orbit quite close to a non-rotating black hole. The dotted circle represents the event horizon and the orbit's closest approach is at $r_p/\bar{M} = 5$, or $2\frac{1}{2}$ times the event horizon radius. The orbit parameters $\tilde{E}, \tilde{L}$ have been chosen so that the periastron shift is 2π radians per orbit, causing the orbit to repeat itself after making a double loop. For the vast majority of values $\tilde{E}, \tilde{L}$, relativistic orbits will not repeat themselves, due to periastron shifts other than 2π, or multiples thereof.

There are more sophisticated methods for solving the equation numerically, but this one is particularly simple, although it is mainly of pedagogical value.

In real life, orbits tend to be even more complex, with multiple sources of gravitation acting upon the orbiting body.

Further, most gravitational sources will be moving relative to whatever reference frame one chooses. Such cases must be solved by perturbation methods or point by point approximations.

Orbital equation for light The Schwarzschild metric, arranged as a light-like interval ($d\tau = 0$), can be used to find the orbital equation for light. It is however very easy to see how equation 7.9 must be modified for light, where $\tilde{L} \to \infty$.

It means that the term $(GM)^2/(\tilde{L}^2c^2)$ in the equation for material bodies falls away and the orbital equation for light becomes simply

$$\frac{d^2u}{d\phi^2} = -u + 3u^2. \tag{7.14}$$

This equation can be used to obtain the gravitational deflection of light when it passes a massive body, as will be discussed in the next chapter. It is still not a solvable differential equation and successive approximations or numerical methods must be used, depending on the purpose of the 'solution'.

7.1 A quasi-Newtonian, 'poor man's orbit'

It is possible to get a pretty accurate orbital plot from the radial and transverse quasi-Newtonian relativistic acceleration components of chapter 5, i.e.,

$$a_r = \frac{-GM}{r^2}\left(g_{tt} - 3g_{rr}\frac{v_r^2}{c^2} + 2\frac{v_t^2}{c^2}\right)$$

and

$$a_t = \frac{2GM}{r^2}g_{rr}\frac{v_r v_t}{c^2},$$

Now the usual methods employed by engineers when simulating dynamic systems can be used, i.e., determining accelerations along the x, y and z axes of a chosen coordinate system and then numerically integrating for velocities and displacements. For a planar orbit, one needs only a two-dimensional coordinate system.

Working with a small positive angular increment $\Delta\phi$, one implements the following broad algorithm (a detailed algorithm is shown in the box on page 115):

Begin algorithm

For a specific orbital position:

Obtain g_{tt}

Calculate the radial and transverse accelerations

Rotate these accelerations to the coordinate axis through the orbital angle

Integrate for the new x and y velocities and positions

Obtain the new radial distance and new orbital angle

Obtain the new radial and transverse velocities by rotating

through the negative of the new orbital angle

Repeat for the new orbital position

End Algorithm

The algorithm does not fully conserve total energy and angular momentum along an elliptical orbit, i.e., it is not fully 'conservative'. Over a full orbit, the errors tend to mostly cancel out.

The reason for the small error is that a_r and a_t are kept constant over a time interval Δt, while in elliptical orbits, they are continously changing. The errors are positive for half of a full orbit and negative for the other half. The 'non-conservative' nature of the algorithm will however cause the orbit to eventually diverge away from a fully 'conservative' orbit.

This aside, the 'poor man's orbit' does remarkably well. In comparison to the more accurate numerical solution of equation 7.9, the error per revolution is very small, visible as a slight anomalous peripsis shift over many revolutions. The error is a function of the time increment per cycle—the smaller the more accurate, at the expense of simulation time, of course.

The test orbit's starting parameters were as follows: $r = 5GM/c^2$, $\phi = 0$, $\Delta\phi = \pi/100000 \approx 30\mu$rad, $v_t = 0.4675c$, $v_r = 0$, giving $\tilde{L} = 3.785GM/c$, and $r_{max} = 26.787GM/c^2$.

The orbit is the same one as pictured in figure 7.1 on page 105, with 360 degrees periapsis shift per orbit. Apart from the slight periapsis shift error, the shape of the orbit is the same as obtained through the more accurate algorithm. The author likes to think that engineers will relate more readily to this algorithm than to the more formal one.

For one thing, it is hard to think of a better way of illustrating the apparent opposing acceleration in the transverse direction. Switch that acceleration off in the simulation and the orbit shape changes dramatically.

There is just no way that a purely radial acceleration can produce an orbit that even approaches the 'real thing', at least not in the strong field, high velocity environment close to a black hole.

For the reader who might want to test out the algorithm on a computer, a few notes. GM/c^2, Δt and r must all be in metres. The author used a constant angular movement per cycle, simply because it speeds up the plot when far from the mass, where the errors are smaller due to the lower velocities. Lastly, GM/c^2 is actually just a scale factor in the plot and can just as well be chosen as unity.

The algorithm is very useful in studying the order of magnitude of the opposing acceleration effects. Apart from being able to view the orbital changes, it is relatively easy to (numerically) extract the values of v_r and v_t anywhere along the orbit.

In the orbit of planet Mercury, the maximum value of $v_r v_t$ is of the order 10^{-8}. With $2GM/c^2/r^2$ of the order 10^{-18} m^{-1}, the opposing transverse acceleration is of the order 10^{-26} m^{-1} maximum, and when multiplied by c^2, it translates to only 10^{-9} m/s^2.

This is a really tiny opposing transverse acceleration (some 0.1 nano-g). It can however be shown that it is the main contributor to the perihelion shift, as will be illustrated in the next paragraph.

Use the test orbit above, giving a periapsis shift of 2π radians and then suppress the opposing transverse acceleration. The algorithm then produces a periapsis shift of about $\pi/2$ radians. One can argue that the other $3\pi/2$ radians must have come from the opposing transverse acceleration.

The values of the radial and transverse acceleration components for the test orbit can also be easily found from the simulation. The maximum opposing transverse acceleration is $a_{t(max)} \approx \pm 2.7 \times 10^{-3} GM/c^2$ m^{-1} and the equivalent total radial acceleration at that time is $a_{r(eq)} \approx -2.7 \times 10^{-2} GM/c^2$ m^{-1}.

So the magnitude of the opposing transverse acceleration component reaches some 10% of the magnitude of the radial acceleration component. This ratio gets higher the closer the periapsis is to the gravity generating mass. This 'explains' the huge periapsis shifts and also the high eccentricity of highly relativistic orbits.

One must however always remember that the transverse acceleration is not 'real', in the sense that there is no 'force' in that direction, or in any other direction for that matter. The orbiter simply moves along a spacetime geodesic, following a 'straight' path in curved spacetime.

The distant observer, essentially sitting in (locally) flat spacetime, observes the orbit and may conclude that there is acceleration, both radial and transverse, as shown in the above equations for a_r and a_t.

To conclude this chapter on orbital dynamics, we will briefly examine one of the interesting mysteries of relativistic orbits—how can they work if gravity propagates at the speed of light, as all effects in relativity are forced to do?

7.2 The 'Speed' of Gravity

Up to now, we have worked with small objects orbiting a large mass, permanently at rest at the origin of an inertial coordinate system. The gravitational field was therefore static in the coordinate system and the 'test object' followed a spacetime geodesic that was determined by the static curvature of spacetime at it's location.

What happens if the test particle is replaced by a massive object of comparable size to the mass at the origin? Such situations are common in the universe, particularly in binary star systems, where the two stars orbit around their common centre of gravity.

The easiest way to analyse binary systems is to choose the coordinate system so that the common centre of gravity is permanently at rest at the origin. Now both masses will be moving (and accelerating),* relative to the chosen

*Accelerating in a Newtonian sense; following spacetime geodesics in a relativistic sense.

coordinate system.

So what happens to the gravitational field? It is reasonable to expect that the field must be varying continuously at every point in the coordinate system. According to Newton's orbital theory, the variations in the gravitational field must occur *simultaneously* with the change of the positions of the two bodies.

This means that the effect of the change in position of a mass must 'propagate' at infinite speed to every point in the coordinate system—an effect called 'action at a distance'. Without this assumption, Newton orbits cannot work as they do, meaning they will not be stable.

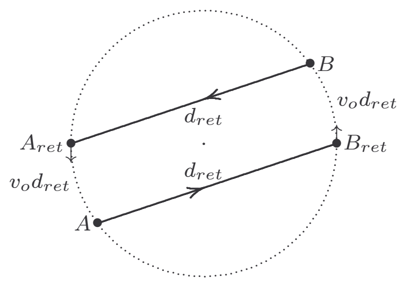

Figure 7.2: Two identical stars in circular orbits around their common centre of gravity (the centre point of the circle). If gravitational forces propagate at the speed of light, then star A will experience a gravitational force towards the 'retarded' position of star B (position B_{ret}) and visa versa for star B. The 'retarded' distance between the stars is indicated by d_{ret}. In the time it takes light to propagate distance d_{ret}, the stars move a distance $v_o d_{ret}$ along the circular orbit, where v_o is the orbital velocity relative to the centre of the circle.

If no effect can propagate faster than the speed of light, as relativity demands, how does a Newton orbit remain stable? Figure 7.2 illustrates the case where the gravitational forces do not point towards the centre of gravity of the two stars, but rather towards the 'retarded' positions of the stars.

The retarded positions is where an observer on each star would have observed the other star, taking into account the propagation delay of light over the distance d_{ret}. This situation does not allow stable orbits in Newton mechanics—there will be residual transverse forces that tend to speed up the orbits and make the two bodies drift apart.

Since we observe stable orbits, we may conclude that nature does not work this way—gravitational effects apparently propagate at a speed of near infinity. The problem is that Einstein's relativity theory forbids any effect to propagate faster than the speed of light—and that includes gravity.

So how can Einstein's theory of gravity produce stable orbits? Further, since Einstein's gravity reduces to Newton's gravity in the limiting case of low fields and low velocity, how can the two theories be reconciled?

The answer lurks in the depth's of solutions to Einstein's field equations,*

> *For a brief, but excellent technical treatment, search the Internet for 'The speed of gravity revisited' [Ibison].

which are unfortunately outside of the scope of this book. Loosely stated, it tells us that for any mass that moves uniformly relative to an inertial frame, (i.e., a non-accelerated mass), it's gravitational field appears static relative to the mass itself—i.e., it moves as if attached to the mass.

A test particle that is kept stationary in the inertial frame and than released, will immediately start to fall towards the proper position of the moving mass and not towards it's retarded position. Now this is totally unremarkable, because we could just as well have viewed the mass as stationary and the test particle as moving relative to the mass. The gravitational acceleration of such a particle would surely be towards the proper position of the stationary mass.

Further, general relativity also tells us that if we could somehow abruptly stop the movement of the mass relative to the inertial frame, the moving gravitational field will also stop moving, but the 'stop effect' will propagate at the speed of light from the mass outwards.

This means that the field will be deformed for a period of time, with some (outer) parts of the field still moving at the original speed and some (inner) parts having already stopped moving. A test particle at some distance from the mass will continue to be curved towards the "extrapolated" position of the moving mass until such time as the change in the gravitational field reaches the particle.

At that time the particle will start to curve towards the proper position of the now stationary mass. Figure 7.3 illustrates the situation in a simplified way.

If the above still sounds weird, take comfort from the fact that Maxwell's equations for electromagnetic radiation predicts exactly this behavior for the field around a moving charge. If the moving charge is abruptly stopped, the field at some measurement point keeps pointing towards the extrapolated position of the charge until the effect of the change in the velocity of the charge has had time to propagate towards the measurement point. This effect has been measured in the laboratory.

The foregoing is readily comprehensible for a uniformly moving mass, but what about the two stars in orbit around their common centre of gravity? Surely, in the inertial reference frame, both masses are constantly being

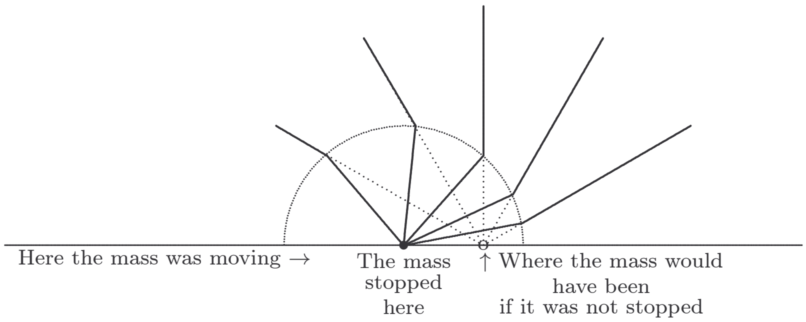

Figure 7.3: A mass was moving uniformly relative to an inertial frame, but is abruptly stopped at time $t = 0$. The effect of the change in velocity of the mass propagates radially outwards at the speed of light (a sphere with radius $c \times t$). Particles outside of the sphere at any given time experience gravity towards the position where the mass would have been if it was not stopped. Particles inside the sphere will experience gravity directly towards the now stationary mass.

accelerated towards their common centre of gravity.

But do the stars experience acceleration in the sense that a force is acting upon them? Not quite.* The two stars are in free-fall and are for all

*The only external force that the stars will experience is tidal gravity, which is the subject of a later chapter.

practical purposes inertially moving masses. More technically correct, they are moving along space-time geodesics, which can be thought of as moving in straight lines through curved space-time.

Their respective gravitational fields are moving with them, just as in the inertially moving mass in the absence of gravity. Relative to their space-time geodesics they experience no acceleration, so their gravitational fields are not deformed, except for small higher order effects that will be discussed below.

If we consider one mass as the 'test particle', that mass will always curve directly towards the proper position of the other mass and not towards it's retarded position. If the stars have sufficient transverse velocity, they will stay in stable orbits around each other.

The above is an oversimplification of the real situation, because general relativity also tells us that there is a higher order interaction between the gravitational fields of the two stars. This interaction causes some deformation of the gravitational fields of both stars and the deformation continuously robs them of a little orbital energy.

For normal binary stars the effect is negligible because of the relatively large distance between them. For binary neutron stars that are very massive and very close to each other, the effect becomes appreciable.

This loss of orbital energy is radiated as gravitational waves, which are the subject of a later chapter. The effect is observable when the binary neutron stars also happen to be pulsars. The orbital periods of the binary pulsars can be deduced very accurately from their highly stable 'light house' effect.

The rate of orbital decay agrees with Einstein's theory to within 1%, which is the experimental uncertainty. Alternatively stated, the binary pulsars tell us that the 'speed of gravity' is within 1% of the speed of light. Einstein's theory of gravity says that gravitational waves and other gravitational disturbances propagate at exactly the speed of light. So it seems that Einstein was right again!

In summary then, because of the 'static' nature of the gravitational field of an inertially moving mass, gravity has the *appearance* of instantaneous propagation. However, static gravitational fields do not propagate, just as static electromagnetic fields do not propagate. Any change to the static gravitational field that is caused by the acceleration* or the deformation

*Acceleration here means being forced out of it's geodesic path through spacetime.

of a gravity generating mass, propagates at the speed of light and so re-establish the changed field.

Orbiting bodies tend to follow space-time geodesics and therefore do not suffer acceleration in the usual sense of the word. Their gravitational fields follow them in their orbits as if they were rigid extensions of the orbiting body.

Where there are more than one mass in a system that generate gravitational fields, the fields do interfere with each other in a way that radiates some of the orbital energy of the system away as gravitational waves.

Algorithm for finding the periapsis and apoapsis for a closed relativistic orbit around a black hole

We have to find the solution to the equation $u^3 + au^2 + bu + k = 0$
Declare all parameters used as double precision floating point values, give conventional (SI) values to G, M, c, $\tilde{E}$, $\tilde{L}$ and calculate the following:

$$\begin{aligned}
\bar{M} &= GM/c^2 \quad \text{`geometrize (to units metres)} \\
\tilde{L} &= \tilde{L}/c \quad \text{`geometrize (to units metres)} \\
\tilde{E} &= \tilde{E}/c^2 \quad \text{`geometrize (dimensionless)} \\
a &= -0.5 \\
b &= \bar{M}^2/\tilde{L}^2 \\
k &= 0.5(\tilde{E}^2 - 1)\ b
\end{aligned}$$

Begin algorithm

$$\begin{aligned}
Q &= (3b - a^2)/9 \\
R &= (9ab - 27k - 2a^3)/54 \\
D &= Q^3 + R^2 \quad \text{`note the } Q\text{-cubed, not squared!} \\
\text{If} \quad D &< 0 \text{ Then} \quad \text{`solution is possible} \\
\theta &= \cos^{-1}(R/\sqrt{-Q^3}) \\
u_1 &= 2\sqrt{-Q}\ \cos\left(\frac{\theta}{3}\right) - \frac{a}{3} \\
u_2 &= 2\sqrt{-Q}\ \cos\left(\frac{\theta + 2\pi}{3}\right) - \frac{a}{3} \\
u_3 &= 2\sqrt{-Q}\ \cos\left(\frac{\theta + 4\pi}{3}\right) - \frac{a}{3} \\
r_1 &= \bar{M}/u_1 \\
r_2 &= \bar{M}/u_2 \\
r_3 &= \bar{M}/u_3
\end{aligned}$$

Else ‘no solution is possible

End If

End algorithm

If a solution is found, the distance r_3 represents the periapsis. If the line $\tilde{E}$ = constant cuts the effective potential curve in three places (i.e., a closed orbit), r_2 represents the apoapsis and r_1 the point to the left of the ‘hump’. If $\tilde{E} = 1$, i.e., escape energy, r_2 will be infinite ($u_2 = 0$) and care must be taken if the algorithm is run on a computer.

Numerical solution of the second order orbital equation

The algorithm is shown in a sort of 'pseudo-BASIC', where $x_n = x_{n-1} + \Delta x$ is written simply as $x = x + \Delta x$. We start the orbit at the periastron or apastron of the orbit, where we set $\phi = 0$.

Declare all parameters used as double precision floating point values Give conventional (SI) values to $r,\ G,\ M,\ c,\ \Delta\phi,\ \tilde{L}$ and then pre-calculate the following:

$$\begin{aligned}
\bar{M} &= GM/c^2 && \text{'geometrize (to units metres)} \\
\tilde{L} &= \tilde{L}/c && \text{'geometrize (to units metres)} \\
u &= \bar{M}/r \\
\frac{d^2u}{d\phi^2} &= \frac{\bar{M}^2}{\tilde{L}^2} - u + 3u^2 \\
\Delta u &= \tfrac{1}{2}\frac{d^2u}{d\phi^2} && \text{'the starting value } \Delta u_{n-1}
\end{aligned}$$

Do Until some condition or user interaction ends loop

$$\begin{aligned}
x &= r\cos\phi, \quad y = r\sin\phi, \quad \text{plot } x \text{ and } y \\
\text{If } \Delta u &<> \frac{-d^2u}{d\phi^2}\Delta\phi^2 \ \text{ Then } \ \text{'normal procedure} \\
\Delta u &= \Delta u + \frac{d^2u}{d\phi^2}\Delta\phi^2 \\
\text{Else} & \qquad \text{'guard agains lock up} \\
\Delta u &= 1E - 99 \times \text{Sgn}(\Delta u) \quad \text{'a tiny nudge} \\
\text{End If} & \\
u &= u + \Delta u \\
\phi &= \phi + \Delta\phi \\
\frac{d^2u}{d\phi^2} &= \frac{\bar{M}^2}{\tilde{L}^2} - u + 3u^2 \\
r &= \bar{M}/u \\
\text{If } r &<= 2\bar{M} \ \text{ Then Exit Do 'falling into black hole}
\end{aligned}$$

Loop

When the term $\frac{\bar{M}^2}{\tilde{L}^2}$ is left out in both calculations of $d^2u/d\phi^2$ above, the 'orbit' of light is obtained.

This algorithm will produce a Newtonian orbit if the term $3u^2$ is left out in both calculations of $d^2u/d\phi^2$ above and $\tilde{L}$ is chosen appropriately (Newtonian).

If both $\frac{\bar{M}^2}{\tilde{L}^2}$ and $3u^2$ are left out, the result is light moving in a straight line, which is what pure Newton mechanics predicts.

Quasi-Newtonian relativistic orbit algorithm

The algorithm is shown in a sort of 'pseudo-BASIC', where $x_n = x_{n-1} + \Delta x$ is written simply as $x = x + \Delta x$.

Declare all parameters used as double precision floating point values. Initialize $\Delta\phi$, G, M, c, r, ϕ, v_r, v_t with starting values and then pre-calculate:

$v_x = v_r \cos\phi - v_t \sin\phi$
$v_y = v_r \sin\phi + v_t \cos\phi$

Do Until some condition or user interaction ends loop

$$
\begin{aligned}
\Delta t &= r\Delta\phi/vt \quad \text{'constant angular rate} \\
g_{tt} &= \sqrt{1 - 2GM/(rc^2)} \\
g_{rr} &= 1/g_{tt} \\
x &= r\cos\phi, \quad y = r\sin\phi \\
& \quad (\text{plot } x \text{ and } y) \\
a_r &= -GM/r^2 \ (g_{tt} - 3g_{rr}v_r^2/c^2 + 2v_t^2/c^2) \\
a_t &= 2GM/r^2 \ g_{rr}v_r v_t/c^2 \\
a_x &= a_r\cos\phi - a_t\sin\phi \\
a_y &= a_r\sin\phi + a_t\cos\phi \\
x &= x + v_x\Delta t + 0.5a_x\Delta t^2 \\
y &= y + v_y\Delta t + 0.5a_y\Delta t^2 \\
v_x &= v_x + a_x\Delta t \\
v_y &= v_y + a_y\Delta t \\
r &= \sqrt{x^2 + y^2} \\
\text{If } r &<= 2GM/c^2 \ \text{ Then Exit Do 'falling through event horizon} \\
\phi &= \phi + \Delta\phi \\
v_r &= v_x\cos\phi + v_y\sin\phi \\
v_t &= -v_x\sin\phi + v_y\cos\phi
\end{aligned}
$$

Loop

Comparison to Faber and MTW orbital equations

For readers familiar with [Faber] and [MTW], a few words of comparison (note that in contrast with this text, both use geometric units throughout). Faber defines $u = 1/r$ (units m^{-1}), so that $du/d\phi = -(1/r^2)dr/d\phi$, also m^{-1}. Just before Faber's equation (163), the following first order differential equation is derived:

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{b^2-1}{h^2} + 2\frac{Mu}{h^2} + 2Mu^3, \qquad (7.15)$$

with M being geometrized as GM/c^2 (units metres), $b = \tilde{E}$ and $h = \tilde{L}$, as symbolized in this text (the MTW symbols).

MTW uses the same definition for M and defines $u = M/r$ (dimensionless), so that $du/d\phi = -(M/r^2)dr/d\phi$, also dimensionless and hence obtains the equivalent equation

$$\left(\frac{du}{d\phi}\right)^2 = \frac{\tilde{E}^2 - (1-2u)(1+L^{\dagger 2}u^2)}{L^{\dagger 2}}, \qquad (7.16)$$

where $L^\dagger = \tilde{L}/M$, a dimensionless momentum parameter. If $L^\dagger = \tilde{L}/M$ is substituted and the equation expanded, the result is

$$\left(\frac{du}{d\phi}\right)^2 = \frac{M^2}{\tilde{L}^2}(\tilde{E}^2-1) + 2\frac{M^2}{\tilde{L}^2}u - u^2 + 2u^3, \qquad (7.17)$$

similar to eq 7.8 of this text (just the units differ). This equation is readily differentiable, giving the second order equation

$$\frac{d^2u}{d\phi^2} = \frac{M^2}{\tilde{L}^2} - u + 3u^2, \quad \text{(dimensionless)}, \qquad (7.18)$$

MTW does not differentiate equation 7.16, hence the more compact form that it was left in. The Faber equivalent of equation 7.18 is (eq. 163)

$$\frac{d^2u}{d\phi^2} + u = \frac{M}{h^2} + 3Mu^2, \quad (\text{units m}^{-1}), \qquad (7.19)$$

where $u = 1/r$, $M = GM/c^2$ and $h = \tilde{L}$ (unit metres). The differences caused by the alternative definitions of u are small, yet subtle. It makes no difference in any calculations, but may be confusing at times. A good question is: why bother with u at all, i.e., why not use r straightaway? The answer is that while eq. (7.17) is third order in u, reducing to second order after differentiation, it would have started as fourth order in r, making life more difficult, mathematically speaking.