Chapter 6

Orbital dynamics - an Introduction

From Newtonian
to
'Einsteinian' orbits

The orbit is an extremely important aspect of gravitational theory, because just about every object that is observable from Earth is in some form of orbit relative to something. The proverbial 'Newton apple' that falls from the tree is attempting to go into orbit around the centre of the Earth. It is however prevented from doing so because the surface of the Earth intervenes.

Any object that is in free-fall in friction-free space obeys two fundamental laws of physics: the conservation of total energy and of total angular momentum. In Newton mechanics, total energy is the sum of potential energy and the kinetic energy of movement and the two types of energy can be exchanged, as long as the sum remains constant.

Angular momentum is the sum of any rotational momentum that the object may have and the angular momentum relative to the origin of whatever coordinate system is used for the measurement. In the case of an apple hanging from a tree, it has angular momentum proportional to the product of it's mass, the speed at which the rotation of the Earth carries it along (relative to the centre of the Earth) and it's distance from the centre of the Earth.

As the apple falls from the tree, it's distance from the centre of the Earth decreases and so does it's potential energy. To compensate, the apple must pick up more speed so that an increase in kinetic energy can make up for the loss of potential energy—this is high school stuff.

Not so well known is the fact that the transverse speed of the apple must also increase, in order to maintain angular momentum as the distance from the centre of the Earth decreases. Surprisingly, this means that the apple does not fall precisely vertically, but will hit the ground a minute distance to the east of the point vertically below it's starting point!*

*Everywhere except at the poles of course!

If the Earth could be reduced to a point mass, the apple would have gone into an elliptical orbit around that point mass—at least if we ignore atmospheric effects. That is how Newton would have argued, based upon his theory of gravity.

We will now use the two conserved quantities to analyse orbits around an isolated, spherically symmetrical and homogeneous massive object, which simplify things considerably. To lay a foundation, we will first consider orbits in Newton mechanics and then show the essential elements of relativistic orbits.

Since many parameters of Newtonian orbits differ from the more general parameters of general relativity, the subscripted N will be used to indicate Newtonian parameters, e.g., E_N indicates Newtonian energy. The unsubscripted parameters with the same name will indicate general relativistic parameters.

6.1 Newtonian orbits

In Newton dynamics, the total orbital energy of a small mass m (the 'test object'), orbiting at a (variable) distance r from the centre of a large mass GM is simply the kinetic energy plus the (negative) potential energy:

$$E_N = \tfrac{1}{2}m(v_r^2 + v_t^2) - GmM/r,$$

where v_r and v_t are the radial and transverse components of orbital velocity, as we used them before. In general, r, v_r and v_t will be constantly changing, but E_N will remain constant.

The second conserved quantity, the angular momentum of the orbit, is given by

$$L_N = m\ r\ v_t,$$

dependant only on the transverse speed component of the orbit (and of course the distance and the mass of the test object).

Engineers will probably more readily recognize the angular momentum as $L = I\ \omega = m\ r^2\ \omega$, where I is the moment of inertia and ω the angular speed. It is the same thing as $L_N = m\ r\ v_t$.

We will later show that this is the equivalent of Kepler's second law, stating that the orbit sweeps out equal areas in equal intervals of time. If the total energy and the angular momentum of a Newton orbit are known, the orbit can be solved.

In orbital analysis it is convenient to work as "normalized as possible". The trick is to get rid of the mass of the orbiting body (m) and work with the total energy parameter (total energy per unit orbiting mass) of the orbiting object, $\tilde{E}_N = E_N/m$, and likewise with the angular momentum parameter, $\tilde{L}_N = L_N/m$, i.e.

$$\tilde{E}_N = \tfrac{1}{2}(v_r^2 + v_t^2) - GM/r \quad \text{and} \quad \tilde{L}_N = r\ v_t, \tag{6.1}$$

where $\tilde{E}_N$ has the dimensions m^2/s^2 and $\tilde{L}_N$ has the units of m^2/s.

Some textbooks, e.g., [MTW], sometimes normalizes the angular momentum parameter further: $L^\dagger = \tilde{L}/M = L/(mM)$, i.e., the angular momentum per unit orbiting mass per unit primary mass. While it makes many orbital equation a bit more compact, it is somewhat 'ugly' looking and some of the intuitive meaning of the equations are lost.

Too much normalization may not always be a good thing, so we will continue to use $\tilde{L}$. It does mean that the fraction $\tilde{L}/M$ occurs frequently in the equations that follow.

Another very useful orbit parameter is the *effective potential*, which is the sum of the *transverse* kinetic energy plus the potential energy, both per unit orbiting mass, i.e.

$$\tilde{V}_N = \tfrac{1}{2}v_t^2 - GM/r,$$

with the obvious units of speed squared, m^2/s^2. By substituting $v_t = \tilde{L}_N/r$, we can write the equation as

$$\tilde{V}_N = \tfrac{1}{2}\tilde{L}_N^2/r^2 - GM/r, \tag{6.2}$$

i.e., a function of the constant $\tilde{L}_N$ and the variable radial distance r.

The physical meaning of $\tilde{V}_N$ is the total energy (parameter) at the 'turning points' of the orbit radius r, i.e., the periapsis and apoapsis. At these points, the radial component of the orbit velocity (v_r) is zero and the total energy is made up only of transverse kinetic energy and potential energy.

By plotting $\tilde{V}_N$ for a specific $\tilde{L}_N$, the turning points (if any) for a given total orbital energy $\tilde{E}$ can be easily found, as illustrated in figure 6.1.

The curve can be understood by noting that at values of $r \gg \tilde{L}_N$, the term $\frac{1}{2}\tilde{L}_N^2/r^2$ becomes negligible and the term $-GM/r$ dominates. At $r = \tilde{L}_N^2/GM$, the curve reaches a minimum and for smaller distances the term $\frac{1}{2}\tilde{L}_N^2/r^2$ starts to become dominant.

The distance $r = \tilde{L}_N^2/GM$ is also where circular orbits will occur (point $\mathbf{A}$ in the figure). Circular orbits represent the minimum total energy for a given angular momentum $\tilde{L}_N$. If the total energy is increased (with constant $\tilde{L}_N$), the orbit becomes elliptical and has a minimum radius at $\mathbf{B}$, with a maximum radius at $\mathbf{B}'$.

A total energy of precisely zero represents escape energy and the orbit will become open (a parabola). All positive total energy levels represent open hyperbolic orbits, at least in Newtonian dynamics.

Another way to look at the effective potential for an elliptical orbit is as follows. For any point along the orbit, the (constant) total orbital energy

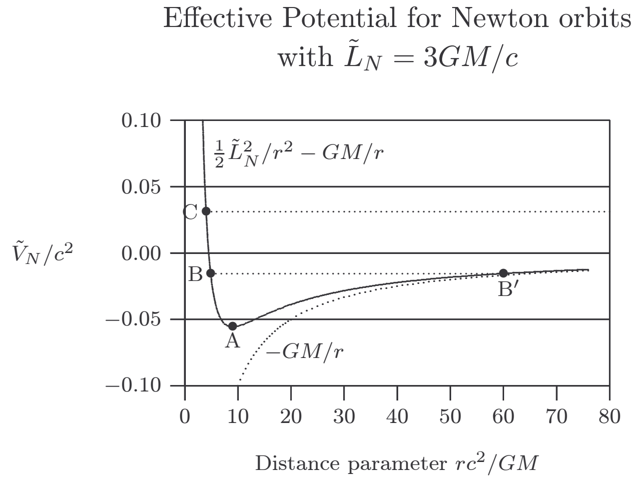

Figure 6.1: The effective potential curve (top, solid) and the potential energy curve (bottom, dotted), drawn in geometric (dimensionless) units. Point **A** represents a circular orbit. Point **B** is the closest approach of a closed, elliptical orbit and **B′** the furthest point, at the same energy level value as point **B**. Point **C** is at positive total energy, which represents an open orbit that will let the orbiting object escape.

is made up of three components: the negative potential energy (dotted curve), the transverse kinetic energy (difference between the sold curve and the dotted curve) and the radial kinetic energy (difference between line **B** - **B′** and the solid curve).

6.2 Kepler's laws for planetary motion

Johannes Kepler established his three laws of planetary orbits in the early 1600s. His first and second laws (elliptic orbits and 'sweeping equal areas in equal time') was established in 1609 and his third law (the correlation between the planetary period and the semi-major axes of the ellipse), followed about ten years later.

Kepler's first law states that a planetary orbit is an ellipse, with the Sun at one of the two foci. The eccentricity (e) of the orbit is related to the orbital constants $\tilde{E}_N$, $\tilde{L}_N$ and the mass M by

$$e = \sqrt{1 + 2\tilde{E}_N \frac{\tilde{L}_N^2}{(GM)^2}}. \tag{6.3}$$

The eccentricity e is less than unity for negative $\tilde{E}_N$, representing closed (circular or elliptical) orbits. When $\tilde{E}_N = 0$, $e = 1$, representing a parabola. For positive $\tilde{E}_N$, $e > 1$ and the orbit is hyperbolic. The orbital equation for (Keplerian) planetary orbits is

$$\frac{r}{GM} = \frac{\tilde{L}_N^2}{(GM)^2} \frac{1}{1 + e\cos\phi}, \tag{6.4}$$

where r is the radial distance from the Sun and ϕ the angular displacement from the point of closest approach along the planar orbit (the perihelion).

By setting $\phi = 0$ and $\phi = \pi$ respectively, the two turning points of the elliptical orbit (the perihelion r_p and aphelion r_a) can easily be found (see figure 6.2).

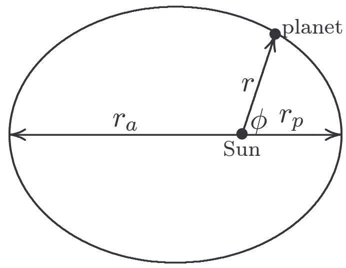

Figure 6.2: An elliptical orbit, showing the relationship between r, r_p, r_a and ϕ. In the case of planetary orbits around the Sun, the perihelion distance r_p occurs when $\phi = 0$ and the aphelion distance r_a when $\phi = \pi$.

Kepler's second law states that a planetary orbit sweeps out equal areas around the Sun in equal time intervals. Let the orbital angle change by $d\phi$, forming an approximate triangle with a 'base' of $rd\phi$ and a 'height' of r. Since $d\phi \to 0$, the area of the triangle (half base times height) is

$$dA = \frac{1}{2}r^2 d\phi = \frac{1}{2}\tilde{L}_N dt.$$

Because $\tilde{L}_N$ is a constant by definition,

$$dA/dt = \frac{1}{2}\tilde{L}_N$$

is also a constant, in agreement with Kepler's second law.

Kepler's third law states that the orbital period of a planet is proportional to the three-halves power of the semi-major axis of the orbit. This can be shown by utilizing Kepler's second law.

The total area of an ellipse is $A = \pi a^2\sqrt{1-e^2}$, where a is the semi-major axis. Divide this area A by Kepler's constant rate $dA/dt = \frac{1}{2}\tilde{L}_N$ and we have the planet's period as

$$T = \frac{2\pi a^2\sqrt{1-e^2}}{\tilde{L}_N}.$$

It can be shown (e.g., [Faber]) that $\sqrt{1-e^2} = \tilde{L}_N/\sqrt{GMa}$, so that

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM}}, \qquad (6.5)$$

which agrees with Kepler's third law.

This relationship is often used to find the mass of the primary object, where a secondary body (planet or a moon) with a known elliptical orbit and a known period is observed. M is easily extracted in terms of the known parameters. It is just as easy to find the semi-major axis a from a known primary mass and a known period.

If the ellipticity is also known, the perihelion and aphelion are found from

$$r_p = a(1-e), \quad \text{and likewise,} \quad r_a = a(1+e).$$

The value of $\tilde{L}_N$ can then be obtained from the orbital equation (6.4), by making $\phi = 0$, i.e.,

$$\tilde{L}_N = \sqrt{r_p(1+e)GM}. \tag{6.6}$$

Then, knowing $\tilde{L}_N$, the effective potential $\tilde{V}_N$ is obtained from equation 6.2, which equals the total energy $\tilde{E}$ for that point (the perihelion).

This method was used to calculate some of the orbital parameters for planet Mercury, with only the period T and the eccentricity e known (together with the mass of the Sun, of course). The results are shown in table 6.1. To get the absolute values of the orbital energy and angular momentum,

Data for planet Mercury				
Parameter	geometric		SI	
Inputs				
mass of Sun M	1.4765×10^{3}	m	1.9891×10^{30}	kg
mass of Mercury m	2.4518×10^{-4}	m	3.3030×10^{23}	kg
period T	2.2786×10^{15}	m	7.6005×10^{6}	s
eccentricity e	0.2056		0.2056	
Calculated				
semi-major axis a	5.7907×10^{10}	m	5.7907×10^{10}	m
perihelion distance r_p	4.6002×10^{10}	m	4.6002×10^{10}	m
aphelion distance r_a	6.9813×10^{10}	m	6.9813×10^{10}	m
angular momentum L_N	2.2186×10^{3}	m^2	8.9605×10^{38}	$\mathrm{kg\ m^2/s}$
angular mom. $\tilde{L}_N = L_N/m$	9.0491×10^{6}	m	2.7128×10^{15}	$\mathrm{m^2/s}$
total energy E_N	-3.1257×10^{-12}	m	-3.7846×10^{32}	$\mathrm{kg\ m^2/s^2}$
total energy $\tilde{E}_N = E_N/m$	-1.2749×10^{-08}		-1.1458×10^{9}	$\mathrm{m^2/s^2}$
velocity at perihelion v_p	1.9671×10^{-4}		58,973	m/s
velocity at aphelion v_a	1.2962×10^{-4}		38,859	m/s

Table 6.1: Some parameters for the planet Mercury, as calculated from the formulae given in the text. The input values were taken from [Mitton], using constants $G = 6.6714 \times 10^{-11}\ \mathrm{m^3 kg^{-1} s^{-2}}$ and $c = 2.9979 \times 10^{8}\ \mathrm{m/s}$.

($E_N = \tilde{E}_N \times m$ and $L_N = \tilde{L}_N \times m$), the mass of Mercury is also needed.

Armed with this information (on what make Newtonian orbits 'tick'), it is time to enter the more formidable arena of relativistic orbits.

6.3 Relativistic orbits

The orbits of general relativity can be solved via the Schwarzschild metric and Einstein's equations for geodesics in spacetime. The actual geodesic equations fall outside the scope of this book, because one needs MTW's 'temptress', differential geometry! The results distilled out of the geodesic equations for Schwarzschild geometry is however quite simple.

Like in the case of Newtonian orbits, the total energy and total angular momentum are also conserved quantities, but they are quite different from the values in Newton orbits. We will first examine these differences and then turn to the orbital equations.

6.3.1 Conservation of energy and angular momentum

The usual way to find the values of the conserved quantities is to extract the rates of change of the coordinates t and ϕ in terms of the proper time (τ) from Einstein's spacetime geodesic equations (e.g., [MTW] eqs. 25.17 and 25.18). Only the results will be given here. Firstly

$$c^2 \frac{dt}{d\tau} = \frac{\tilde{E}}{g_{tt}},$$

where τ represents proper time, $\tilde{E}$ is the relativistic total energy parameter and $g_{tt} = 1 - 2GM/(rc^2)$, the time-time coefficient of the Schwarzscild metric, giving

$$\tilde{E} = c^2 g_{tt} \frac{dt}{d\tau} = \frac{c^2 g_{tt}}{\sqrt{g_{tt} - g_{rr} v_r^2/c^2 - v_t^2/c^2}}. \tag{6.7}$$

The expansion of $dt/d\tau$ comes from the Schwarzschild metric, v_r, v_t are the radial and transverse velocity components and $g_{rr} = 1/g_{tt}$, as before.

It can be shown that when $v_r, v_t \ll c$ and $r \gg 2GM/(rc^2)$, the expression approximates to the Newton energy parameter, plus the constant c^2, i.e. $\tilde{E} \approx \frac{1}{2}(v_r^2 + v_t^2) - GM/r + c^2$. The added c^2 corresponds to the fact that Einstein adds the rest energy ($E = mc^2$) of an object to it's total energy , while Newton does not. Recall that $\tilde{E}$ represents energy per unit rest mass.

The second conserved quantity, angular momentum, is also obtained from the geodesic equations, i.e.,

$$\frac{d\phi}{d\tau} = \frac{\tilde{L}}{r^2},$$

where $\tilde{L}$ is the relativistic angular momentum parameter, giving

$$\tilde{L} = r^2 \frac{d\phi}{dt} \frac{dt}{d\tau} = \frac{r\ v_t}{\sqrt{g_{tt} - g_{rr} v_r^2/c^2 - v_t^2/c^2}}, \tag{6.8}$$

which approximates to the Newton case in the low gravity, low velocity limit, where $g_{tt}, g_{rr} \to 1$ and $v_r, v_t \ll c$. Recall that the $\tilde{L}$ has the unit m^2/s.

To wrap up the orbital parameters, we can get the relativistic effective potential energy parameter by extracting v_t from eq. 6.8 and substitute it into eq. 6.7 (with $v_r = 0$), giving

$$\tilde{V} = c^2 \sqrt{g_{tt}(1 + \tilde{L}^2/(r^2 c^2))}, \tag{6.9}$$

which is equal to $\tilde{E}$ at the turning points of the orbit. In the low gravity, low angular momentum limit ($g_{tt} \approx 1$ and $\tilde{L}/c \ll r$), the equation approximates to the Newton case (plus c^2), i.e., $\tilde{V} \approx \frac{1}{2}\tilde{L}^2/r^2 - GM/r + c^2$. Figure 6.3 shows the relativistic and the Newton effective potentials on the same scale, with the c^2 added to the Newton case to make it easy to compare.

Newton's and Einstein's Effective Potentials
compared for $\tilde{L} = 4.2GM/c$

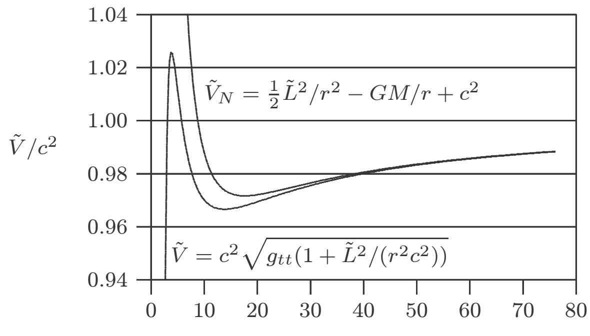

Figure 6.3: The top curve is the Newton effective potential (with c^2 added to make it easy to compare) and the bottom curve the general relativistic effective potential, both drawn for the same $\tilde{L}$. Note how closely the two curves match for distances $rc^2/GM > 50$.

These two curves show the dramatic differences between the effective potentials given by the two theories, at least for the 'near' region. For $rc^2/GM > 50$, the differences get vanishingly smaller (always remembering the c^2 that was added for easy comparison, of course).

In the near region it is clear that, for the same $\tilde{L}$ and $\tilde{V}$, a relativistic orbit's periapsis will generally be closer and the apoapsis (slightly) farther from the black hole than for the Newton case.*

*The fact that we have added c^2 to the Newton effective potential, makes no difference to the turning points.

Generally, one can say that the eccentricity of a relativistic orbit is larger than the equivalent Newton orbit. Further, for a given $\tilde{L}$, circular orbits (at the minima of the curves) are closer to the black hole for relativistic orbits than for Newton's. This comes from the larger gravitational acceleration in relativistic dynamics.

Just like in the case of the Newton effective potential, the relativistic curve can be used to determine broad characteristics of various orbits. The three usual types of orbits for objects with mass, i.e., circular, non-circular (closed) and open, are shown in figure 6.4.

Points **A**, **B** and **C** are the equivalents of the corresponding points in Newton dynamics. At point **D** (the peak of the curve), a circular orbit is theoretically possible, but unlike point **A**, it is unstable and any perturbation will cause the object to either escape or spiral into the black hole.

Orbit turning points in General Relativity
for $\tilde{L} = 4.2GM/c$

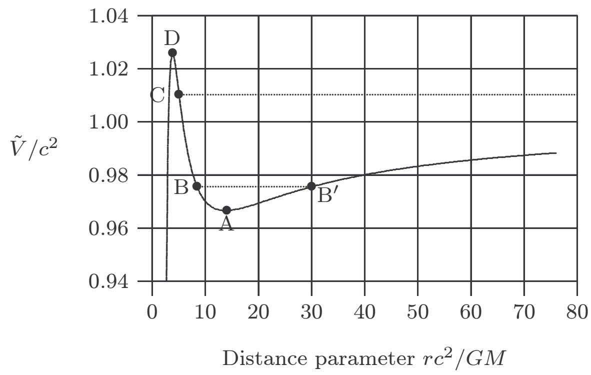

Figure 6.4: Points **A, B** and **C** represent the energies for circular, closed and open orbits respectively, around a Schwarzschild black hole. Point **D**, the maximum of effective potential (for $\tilde{L} = 4.2GM/c$), represents an unstable circular orbit that will either escape or spiral into the hole.

At energy levels larger than point **D**, there is no orbit solution possible and objects will either escape or be 'swallowed' by the hole, depending on the geometry of the orbit. Contrast this with the Newtonian case, where such an object could in principle enter the black hole, swing around the central singularity and then escape again.

In general relativity, no orbiting object without some means of propulsion, can venture as close as $r = 3GM/c^2$ to a Schwarzschild black hole, without being dragged in and eventually being crushed out of existence in the central singularity.

Solution of the orbit's turning points The shape of the effective potential energy curve changes for different values of the angular momentum parameter $\tilde{L}$. Generally speaking, the trough and the peak shifts further apart when $\tilde{L}$ increases and move closer to each other when $\tilde{L}$ decreases—closer both along the energy and the distance axis.

As is clear from figure 6.4, the turning points of an orbit are where $\tilde{V}^2 = \tilde{E}^2$, i.e., where the constant energy line cuts the effective potential curve. For any closed orbit, the line $\tilde{E}^2 =$ constant cuts the effective potential curve in three places.

For open orbits that is either escaping, or falling into the mass, the constant energy line cuts the curve in only two places. The radial distances of the turning points are given by the following third order (or cubic) equation*

*From eq. 7.8 in the next main section, with the radial rate $du/d\phi = 0$.

$$u^3 - u^2/2 + \bar{M}^2 u/\tilde{L}^2 + (\tilde{E}^2 - 1)\bar{M}/(2\tilde{L}^2) = 0,$$

where $u = \bar{M}/r$, $\bar{M} = GM/c^2$ (i.e., geometric units where $G = c = 1$, so that $\bar{M}$ has the units metres and u is dimensionless), all for mathematical convenience and clarity. To find the values, we have to solve a third order equation of the form

$$u^3 + au^2 + bu + k = 0.$$

The algorithm for solving this form is given in most books on mathematical formulae, and it is a bit of a thriller! For those interested, a tailored version is shown in the box at the end of this chapter (page 113).

The important thing is however that relativistic orbits are 'solvable' to some degree. In the latter part of this chapter, we will have a closer look at the general solution of relativistic orbital equations.

The minimum value of effective potential At $\tilde{L} \cong 3.4651GM/c$, the peak coincides with the trough, precisely at $r = 6GM/c^2$. This represents the minimum angular momentum that will allow a stable orbit around a Schwarzschild black hole, as illustrated in figure 6.5.

Since there is no longer a 'hump', more total energy will not save an object from falling into the hole, unless it has enough energy to escape *and* that energy is directed outwards. Otherwise, it will eventually pass too close to the hole and spiral into it. An orbiter must have an angular momentum parameter $\tilde{L} > 3.4651GM/c$ in order to remain in a stable, closed orbit around a black hole.

This is not quite 'rocket science', because an Earth orbiter also needs a minimum angular momentum in order to stay above the atmosphere. If the launch vehicle fails to deliver this angular momentum, e.g., if it delivers too much radial velocity instead of transverse velocity, the satellite will eventually fall back and enter the atmosphere.*

*Unless the total energy is enough to escape, which is not normally the case with satellite launches.

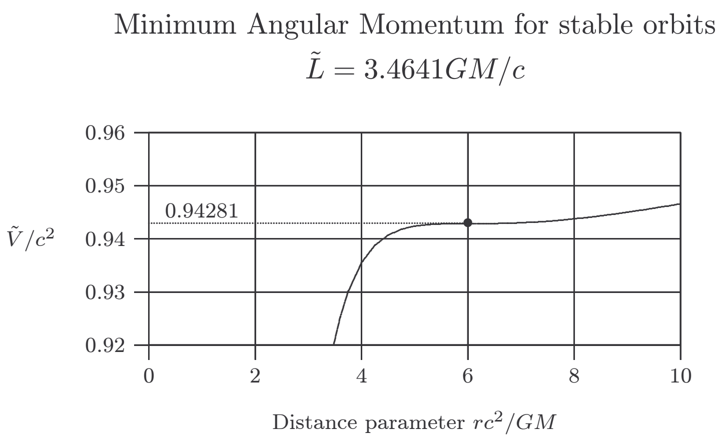

Figure 6.5: With angular momentum parameter $\tilde{L} < 3.4641GM/c$, no circular orbit around a Schwarzschild black hole can be stable, but will spiral into the hole. This also means that circular orbits with $r < 6GM/c^2$, or with total energy $\tilde{E} < 0.94281c^2$ are unstable for such a black hole.

Elliptical orbits can venture closer to the hole than $r = 6GM/c^2$, provided that they have enough angular momentum to create a 'hump' in the effective potential curve—and provided that their total energy parameter $\tilde{E}/c^2$ is not larger than the amplitude of the 'hump'.

It seems that a reasonably 'safe' angular momentum is one that creates a 'hump' amplitude $\tilde{V}/c^2 = 1$; at least an orbiting object will then escape if the total energy parameter is larger than the hump. This is (again) provided that the radial component of the energy is directed outwards.

It is fairly easy to show that $\tilde{L} = 4GM/c$ provides just that—a 'hump' amplitude equal to the escape energy. This $\tilde{L}$ curve has a trough at $r = 12GM/c^2$, which can be considered the 'ultra safe' circular orbit radius around a Schwarzschild black hole.

If something large hit's your ship and imparts a lot of energy on it, chances are pretty good that the ship will escape from the hole. It is only if the collision imparts a lot of radially inwards energy on your ship that you are still in trouble. But then, if you had the propulsion system to take you to a black hole in the first place, that system should be good enough to save your skin.

A spaceship with a serious propulsion system can venture quite close to a Schwarzschild black hole. This is because it can manage quite different dynamics than what a purely free-falling object has at it's disposal.

However, near a black hole, the dynamics become very interesting. Suppose you have a spaceship in circular orbit at a reasonably safe distance (say at $r = 7GM/c^2$) from a rather large black hole. By activating a series of short reverse thrust 'burns', you can reduce the ship's orbital energy and momentum with each burn and settle into a stable, yet slightly smaller orbit after each burn.

With a bit of juggling, you can circularize your orbit at each step. That is until you reach the *marginally stable orbit radius* of $r = 6GM/c^2$. You have reached the minimum angular momentum for stable circular orbits, $\tilde{L} = 3.4641GM/c$.

Any further reverse thrust burns will cause the spaceship to spiral inwards, rather than settle into a slightly smaller stable orbit. You will have to continously juggle between reverse thrust and forward thrust burns, in order to slow down the inspiral, because circular orbits are no longer stable.

As the ship nears the limit for *marginally bound orbits*, i.e., $r = 4GM/c^2$, the burns become very critical in terms of exactly how much total energy the ship retains after the burn. A bit too much energy and the ship will rapidly spiral outwards and tend to escape from the hole.

At $r = 4GM/c^2$, the circular orbit velocity equals the escape velocity—hence a pretty unstable situation arises. A bit too little energy and the ship will tend to plunge dangerously inwards. Suppose that, with the aid of the on-board computer, you succeed in achieving a slow inspiral by a precise combination of reverse and forward thrust burns.

The speed of the ship will build up rapidly and as the orbit radius nears the value $r = 3GM/c^2$, the ship will approach the local speed of light. Since it cannot quite reach the speed of light, even forward thrust burns cannot save you and the ship from being dragged into the hole.

The only way out is to arrange the burns so that there is a radially outwards component in the thrust. In fact, any forward thrust component will be a waste of energy, because the orbital speed will remain more or less constant, near the local speed of light. You should rather orientate the ship so that all the thrust is directed radially outward. In fact, it will be better to thrust somewhat backwards and outwards, because the velocity vector of the ship will never exceed the speed of light.

What is more, the engine should be kept running constantly at just the right thrust so that a slow inspiral is achieved. Even this scheme is full of danger, because controlling a spaceship orbiting at practically the speed of light, so close to a black hole ($r < 3GM/c^2$), is a horrendously complex task.

Before crossing the 'extreme danger radius', you should rather abandon the exercise and blast the ship to a safe orbital radius again. The only 'safe' procedure for getting close to the event horizon of a Schwarzschild black hole is to slowly descent radially from a safe distance and then hover just above the event horizon with the engine blazing.

One complication of this scheme is that, unlike in a free-falling orbit, you and your crew might be subjected to horrendously high 'g-forces', depending on the size of the black hole. Interestingly enough, the more massive the hole, the easier it will be on your bodies, because you will be more distant from it's centre.

There will also be tidal forces that tend to stretch you in the radial direction and squeeze you in transverse directions relative to the hole. We will defer discussion of these forces until chapter 9, where they will be analyzed.

6.3.2 Effective potential of light

The above analysis of orbits does not hold for light, because the 'rest mass' of a photon is zero and the angular momentum parameter is undetermined ($L/m = r^2 d\phi/d\tau$, where both m and $d\tau$ are zero). There are however ways to determine an equivalent 'effective potential' for light.

Both $\tilde{E}$ and $\tilde{L}$ goes to infinity, but the ratio of the two, $b = \tilde{L}/\tilde{E}$ remains finite and is called the *impact parameter* for light. This is however not the effective potential for light, which is defined as (e.g., [MTW])

$$B^{-2} = \frac{g_{tt}}{r^2}c^2, \tag{6.10}$$

plotted in figure 6.6 in normalized form.

One can think of the effective potential of light as the transverse velocity of light divided by the distance. The transverse speed of light (c_{tr}) at Schwarzschild radial distance r is $c_{tr} = \sqrt{g_{tt}}$.

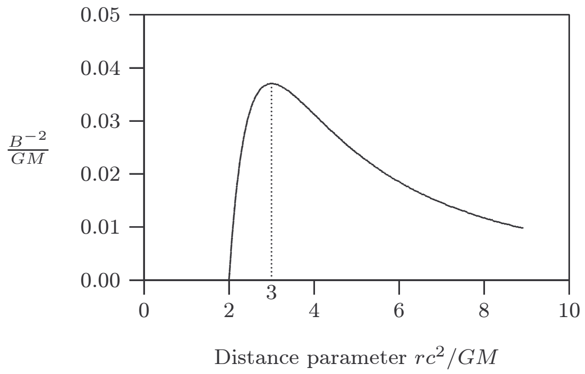

Figure 6.6: The effective potential for light. When the closest approach is $r > 3GM/c^2$, the light will be deflected, but will always escape the black hole. At $r = 3GM/c^2$, the light can be in an unstable circular orbit around the hole, while with $r < 3GM/c^2$, the light will always spiral into the hole.

The angular movement in time dt is $d\theta = c_{tr}dt/r$ and therefore the instantaneous angular velocity at closest approach (when the movement is purely transverse) is

$$\frac{d\theta}{dt} = \frac{\sqrt{g_{tt}}}{r}c = B^{-1}.$$

The effective potential for light, as defined above, is just the square of the angular velocity at closest approach, as plotted in the figure.

The curve has no local minimum value, but tends to zero when either $r \to 2GM/c^2$, or $r \to \infty$. There is only one bound orbit possible, and that is at the maximum, when $r = 3GM/c^2$, but even that is an unstable situation, because any perturbation will cause the light to either escape or fall into the black hole.

Light rays that pass a Schwarzschild black hole at a closest distance of $r > 3GM/c^2$ will be deflected, but cannot be captured. Passing light rays that venture inside $r = 3GM/c^2$ will always spiral into the hole. At the borderline case between the two, there is a change that the light may be captured into a circular orbit.

It is however possible for light rays generated (or reflected from) inside the $2GM/c^2 < r < 3GM/c^2$ zone to escape from a Schwarzschild black hole, but then the light needs a positive radial velocity component. From just outside the Schwarzschild radius ($r = 2GM/c^2$), light only escapes if the initial direction is very close (or equal) to precisely radially outward.

It is shown in [MTW] that, in order for light to escape from $r < 3GM/c^2$, the angle between the propagation direction of the light and the radial direction must satisfy

$$\sin\delta < 3\sqrt{3}\,\frac{GM}{c^2}B^{-1} = 3\sqrt{3}\,\frac{GM}{c^2}\frac{\sqrt{g_{tt}}}{r}.$$

The inverse is also true: all infalling light rays will tend towards precisely radially inward as they near the event horizon. If you hover your spaceship (with engines blazing) just outside the event horizon, all incoming light will come from a narrow cone directly 'overhead'.

The whole 360 degrees of visible universe will be compressed into that cone and the rest will be just black, creating the illusion that the black hole is busy engulfing you—not a nice place to be!

Kip Thorne describes this vividly in [Thorne]. To avoid capture, the condition for incoming light rays that are still outside $r = 3GM/c^2$, is the opposite of the above, i.e.,

$$\sin\delta > 3\sqrt{3}\,\frac{GM}{c^2}\frac{\sqrt{g_{tt}}}{r},$$

meaning that the (inwards) propagation direction must be deviate from the radial by more than δ.

Now that we have a fair idea of the mechanisms of relativistic orbits, we can attempt to also understand the trajectories that material objects and light will follow outside of a Schwarzschild black hole. To understand, we have to turn to the actual orbital equations, which is the subject of the next chapter.