Chapter 11

Parametrized Post-Newtonian formalism

good enough for
many
practical purposes

In the relatively slow motion, weak gravitational fields of the solar system, general relativity can be approximated to something between Newton dynamics and 'Einstein dynamics'. Einstein's field equations are prohibitively hard to solve for multi-body systems like the solar system, even approximately.*

*In fact, rigorous solutions for multi-body problems do not exist in general.

Full solutions are not quite necessary in the weak field, low velocity limit. This is where scientists adopted the engineering-like approach of 'good enough for most (or many) practical purposes'. Eddington, Robertson and Schiff started the movement towards 'post-Newtonian' approximations of general relativity.

Note that, despite the name, 'post-Newtonian' does not mean modified Newton gravity, but rather simplified Einstein gravity. Nordtvedt later expanded on the scheme and it became known as 'Parametrized Post-Newtonian' (PPN) formalism.

It was realized that most post-Newtonian theories of gravity predicted (in the slow moving, weak field limit) a spacetime metric that has a similar structure. Most gravitational theories (including general relativity) could, in this limit, be expressed as an expansion of the Minkowski metric, but with different parameters as coefficients.

The modern, generalized and unified version of PPN formalism is due to Will and Nordtvedt. This chapter is included because if engineers get involved in observational relativity, the PPN formalism is most probably the mathematics that will confront them. The aim of this chapter is merely to introduce the topic, so that further reading can, hopefully, be easier.

11.1 Post-Newtonian approximation

To understand the reasoning behind the post-Newtonian approximation, it is good to revisit the exact Schwarzschild line element for the gravitational field outside a stationary, spherically symmetrical and non-rotating mass $\bar{M}$:

$$ds^2 = -\left(1 - 2\frac{\bar{M}}{r}\right) dt^2 + \left(1 - 2\frac{\bar{M}}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,$$

where θ and ϕ are spherical coordinate angles in two orthogonal directions. The first term represents a time interval and the second term a radial space interval (dr suggests a change in the radial distance r). The last two terms represent transverse space intervals, relative to mass $\bar{M}$.

As stated in previous chapters, movement in the radial and transverse directions encounter different amounts of space curvature. The dr^2 term has a coefficient that is a function of the inverse of the gravitational time dilation, while the transverse terms do not have such a coefficient.

It is this difference between radial and transverse movement that complicates matters significantly when real problems must be solved in Schwarzschild coordinates. To overcome this difficulty, relativists perform a transformation of coordinates that yields a 'conformally flat space' (not flat spacetime) coordinate system, called *isotropic coordinates*, which loosely means 'the same in all space directions'.

The isotropic radial space distance is obtained by the transformation:

$$r = \bar{r}\left(1 + \frac{\bar{M}}{2\bar{r}}\right)^2,$$

where r is the usual Schwarzschild radial distance and $\bar{r}$ is the isotropic radial distance. This transforms the Schwarzschild metric to (e.g., [MTW] exercise 31.7)

$$ds^2 = -\left(\frac{1 - \bar{M}/2\bar{r}}{1 + \bar{M}/2\bar{r}}\right)^2 dt^2 + \left(1 + \frac{\bar{M}}{2\bar{r}}\right)^4 \left(d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2 \theta d\phi^2\right).$$

When comparing this line element to the Schwarzschild line element above, note the following:

$$\left(\frac{1 - \bar{M}/2\bar{r}}{1 + \bar{M}/2\bar{r}}\right)^2 = 1 - 2\bar{M}/r \quad \text{(identically)},$$

although it may not look like it. They operate on the same time parameter dt.

Also note the crucial difference between the isotropic and the Schwarzschild coordinates: the radial space term $(d\bar{r}^2)$ and the transverse space terms $(\bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta d\phi^2)$ are now multiplied by the same coefficient $(1 + \bar{M}/2\bar{r})^4$.

This means that *space* is 'conformally flat' in this metric, meaning space curvature is identical in the radial and transverse directions, simplifying calculations for complex cases significantly. *Spacetime* is however not conformally flat, because the coefficients for time and space are different.

Relativistic gravity in the solar system is usually studied in isotropic coordinates and the relativistic ephemeris for the solar system, drawn up by the Caltech Jet Propulsion Laboratory, uses this coordinate system. So does the PPN formalism, which will be elaborated upon below. Taken all together, the isotropic coordinate system is an important scheme.

As an aside, it also has the added attraction that the metric ds does not diverge to infinity at $r = 2\bar{M}$, or $\bar{r} = \bar{M}/2$. As $\bar{r}$ gets smaller than $\bar{M}/2$, the corresponding Schwarzschild radial distance r increases again and tends to $+\infty$ as $\bar{r} \to 0$, so that isotropic coordinates reach the event horizon, but never enter the black hole. (See figure 11.1.) If $\bar{r} \gg \bar{M}$, then $\bar{r} \approx r - \bar{M}/2$, which is very close to r.

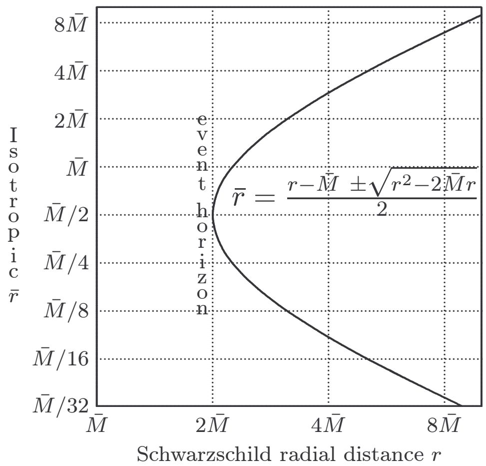

Figure 11.1: A log-log plot of isotropic radial distance $\bar{r}$ as a function of Schwarzschild radial distance r.

What will the gravitational redshift factor look like in isotropic coordinates? It has the value

$$g_{tt} = \sqrt{1 - 2\bar{M}/r} = \frac{|1 - \bar{M}/2\bar{r}|}{1 + \bar{M}/2\bar{r}}.$$

Figure 11.2 shows a plot of g_{tt} agianst $\bar{r}$. Each value of redshift corresponds to two values of $\bar{r}$, but they both represent the same Schwarzschild radial distance r.

Despite the fact that isotropic coordinates offers conformally flat space, the full isotropic line element is still not all that easy to work with. For the type

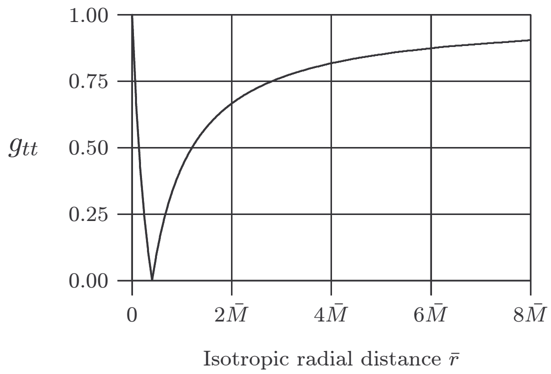

Figure 11.2: A linear plot of the gravitational redshift factor g_{tt} as a function of isotropic radial distance $\bar{r}$, where $g_{(r)} = \frac{|1-\bar{M}/2\bar{r}|}{1+\bar{M}/2\bar{r}}$. For practical situations one can ignore values of $\bar{r} < 0.5$.

of gravitational fields and velocities found in the solar system, higher order terms of $\bar{M}/\bar{r}$ are negligible in the spatial part of the metric (the right-most term) and the isotropic line element is usually approximated to

$$ds^2 \cong -\left[1 - 2\frac{\bar{M}}{\bar{r}} + 2\left(\frac{\bar{M}}{\bar{r}}\right)^2\right] dt^2 + \left(1 + 2\frac{\bar{M}}{\bar{r}}\right)\left(d\bar{r}^2 + \bar{r}^2 d\psi^2\right),$$

called the *first post-Newtonian* (or 1PN) approximation.

The value of $(\bar{M}/\bar{r})^2$ is also extremely small,* but as we will see later,

*For planet Mercury at perihelion, $(\bar{M}/\bar{r})^2 \approx 10^{-15}$, as compared to $\bar{M}/\bar{r} \approx 3\times10^{-8}$.

it plays an important role in distinguishing rival theories of gravity from general relativity.

Since it is no longer necessary to treat radial and transverse spatial intervals differently, the polar space coordinates $[d\bar{r}^2 + \bar{r}^2 d\psi^2]$ can be replaced by the simpler Cartesian values $[dx^2 + dy^2 + dz^2]$, giving

$$ds^2 \cong -\left[1 - 2\frac{\bar{M}}{\bar{r}} + 2\left(\frac{\bar{M}}{\bar{r}}\right)^2\right] dt^2 + \left(1 + 2\frac{\bar{M}}{\bar{r}}\right)\left(dx^2 + dy^2 + dz^2\right),$$

the most common form encountered.

In many books and articles, the bars are dropped from the radial distance $\bar{r}$ and it must then not be confused with the Schwarzschild radial distance r—they are different, yet very close to each other, as we have seen above.

The notations r' or r_i are also sometimes used for isotropic radial distance. This can all be very confusing to laypeople, but relativists are so familiar with their equations, that they read them correctly by simply noting their form.

11.2 Parametrized Post-Newtonian approximation

The post-Newtonian approximation is of a form that allows easy incorporation of other theories of gravity, by means of parameters included in the coefficients of the line element ds. Such schemes are called Parametrized Post-Newtonian (PPN) approximations.

The two most important PPN parameters are β and γ and they enter the above post-Newtonian equation as follows:

$$ds^2 \cong -\left[1 - 2\frac{\bar{M}}{\bar{r}} + 2\beta\left(\frac{\bar{M}}{\bar{r}}\right)^2\right] dt^2 + \left(1 + 2\gamma\frac{\bar{M}}{\bar{r}}\right)\left(dx^2 + dy^2 + dz^2\right),$$

where for general relativity, $\beta = \gamma = 1$.

Loosely speaking, the meaning of β is the amount of non-linearity in the superposition law for gravity, where $\beta = 1$ means no non-linearity. It can also be thought of as the 'equivalence principle parameter'. Theories of gravity that conform to the Einstein equivalence principle all have $\beta = 1$, while those that do not conform to it have $\beta \neq 1$.

The parameter γ represents the amount of space curvature produced by one unit of rest mass, where general relativity is taken to produce one unit of curvature per unit rest mass. There are quite a few theories of gravity that gives a slightly different curvature per unit mass, i.e. $\gamma \neq 1$.

The PPN approximation is not normally stated in terms of the line element ds^2, but the same information is expressed in terms of the metric tensor $g_{\mu\nu}$, where μ, ν = 0,1,2,3, i.e., it is a 4x4 matrix, as shown below (a bit over elaborated, as a 'bordered-matrix', to make the relationship with the line element as clear as possible).

$$(g_{\mu\nu}) = \begin{array}{c} \\ dt \\ dx \\ dy \\ dz \end{array}\begin{array}{c} \begin{array}{cccc} dt & dx & dy & dz \end{array} \\ \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix} \end{array},$$

where

$$\begin{aligned} g_{00} &= -\left[1 - 2\frac{\bar{M}}{\bar{r}} + 2\beta\left(\frac{\bar{M}}{\bar{r}}\right)^2\right], \\ g_{ij} &= \left(1 + 2\gamma\frac{\bar{M}}{\bar{r}}\right)\delta_{ij}. \end{aligned}$$

Indices $i, j = 1, 2, 3$, i.e., this is for space, not spacetime. Only the space coefficients g_{11}, g_{22} and g_{33} are non-zero, as is clear from the matrix.

The factor δ_{ij} is called the 'Kronecker delta', which has the following meaning: if $i = j$, then $\delta_{ij} = 1$; if $i \neq j$, then $\delta_{ij} = 0$.